Chapter 1 from Landau in Naproche

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This is a reformulation of the first chapter of Landau's *Grundlagen der Analysis* in the Controlled Natural Language of Naproche. Talk about sets is still avoided. One consequence of this is that Axiom 5 (the induction axiom) cannot be formulated; instead we use an induction proof method.

Axiom 3: For every x, $x' \neq 1$.

Axiom 4: If $x' = y'$, then $x = y$.

Theorem 1: If $x \neq y$ then $x' \neq y'$.
Proof:
Assume that $x \neq y$ and $x' = y'$. Then by axiom 4, $x = y$. Qed.

Theorem 2: For all x $x' \neq x$.
Proof:
By axiom 3, $1' \neq 1$. Suppose $x' \neq x$. Then by theorem 1, $(x')' \neq x'$. Thus by induction, for all x $x' \neq x$. Qed.

Theorem 3: If $x \neq 1$ then there is a u such that $x = u'$.
Proof:
If $1 \neq 1$ then there is a u such that $1 = u'$.
Assume $x' \neq 1$. If $u = x$ then $x' = u'$. So there is a u such that $x' = u'$.
Thus by induction, if $x \neq 1$ then there is a u such that $x = u'$. Qed.

Definition 1:
Define + recursively:
$x + 1 = x'$.
$x + y' = (x + y)'$.

Theorem 5: For all x, y, z, $(x + y) + z = x + (y + z)$.
Proof:
Fix x, y.
$(x + y) + 1 = (x + y)' = x + y' = x + (y + 1)$.

Assume that $(x+y)+z = x+(y+z)$. Then $(x+y)+z' = ((x+y)+z)' = (x+(y+z))' = x+(y+z)' = x+(y+z')$. So $(x+y)+z' = x+(y+z')$.
Thus by induction, for all z, $(x+y)+z = x+(y+z)$. Qed.

Lemma 4a: For all y, $1+y = y'$.
Proof:
By definition 1, $1+1 = 1'$.
Suppose $1+y = y'$. Then by definition 1, $1+y' = (1+y)'$. So $1+y' = (y')'$.
Thus by induction, for all y $1+y = y'$. Qed.

Lemma 4b: For all x,y, $x'+y = (x+y)'$.
Proof:
Fix x. Then $x'+1 = (x')' = (x+1)'$ by definition 1.
Suppose $x'+y = (x+y)'$. Then by definition 1 $x'+y' = (x'+y)' = ((x+y)')' = (x'+y)'$.
Thus by induction, for all y $x'+y = (x+y)'$. Qed.

Theorem 6: For all y, x, $x+y = y+x$.
Proof:
Fix y. Then $y+1 = y'$. By lemma 4a, $1+y = y'$. So $1+y = y+1$.
Assume that $x+y = y+x$. Then $(x+y)' = (y+x)' = y+x'$. By lemma 4b, $x'+y = (x+y)'$, i.e. $x'+y = y+x'$.
Thus by induction, for all x $x+y = y+x$. Qed.

Theorem 7: For all x, y, $y \neq x+y$.
Proof:
Fix x. Then $1 \neq x'$, i.e. $1 \neq x+1$.
If $y \neq x+y$, then $y' \neq (x+y)'$, i.e. $y' \neq x+y'$.
So by induction, for all y $y \neq x+y$. Qed.

Theorem 8: If $y \neq z$, then for all x $x+y \neq x+z$.
Proof:
Assume $y \neq z$. Then $y' \neq z'$, i.e. $1+y \neq 1+z$.
If $x+y \neq x+z$, then $(x+y)' \neq (x+z)'$, i.e. $x'+y \neq x'+z$.
So by induction, for all x $x+y \neq x+z$. Qed.

Theorem 9: Fix x, y. Then precisely one of the following cases holds:
Case 1: $x = y$.
Case 2: There is a u such that $x = y+u$.
Case 3: There is a v such that $y = x+v$.
Proof: Fix x, y. By theorem 7, case 1 and case 2 are inconsistent and case 1 and case 3 are inconsistent. Suppose case 2 and case 3 hold. Then $x = y+u = (x+v)+u = x+(v+u) = (v+u)+x$.
Contradiction by theorem 7. Thus case 2 and case 3 are inconsistent. Thus for all x, y, at most one of case 1, case 2 and case 3 holds.
Now fix x. Define $M(y)$ iff case 1 or case 2 or case 3 holds.

Suppose $y = 1$. By theorem 3, $x = 1 = y$ or $x = u' = 1 + u = y + u$. Thus $M(1)$.
Suppose $M(y)$. Then there are three cases:
Case 1: $x = y$.
Then $y' = y + 1 = x + 1$. So $M(y')$.
Case 2: $x = y + u$.
If $u = 1$, then $x = y + 1 = y'$, i.e. $M(y')$.
By theorem 3, if $u \neq 1$, then $u = w' = 1+w$, i.e. $x = y+(1+w) = (y+1)+w = y'+w$, i.e. $M(y')$.
Case 3: $y = x + v$.
Then $y' = (x + v)' = x + v'$, i.e. $M(y')$.
So in all cases $M(y')$.
Thus case 1 or case 2 or case 3 holds. Qed.

Definition 2:
Define $x > y$ iff there is a u such that $x = y + u$.

Definition 3:
Define $x < y$ iff there is a v such that $y = x + v$.

Theorem 10: Let x, y be given. Then precisely one of the following cases holds:
$x = y$. $x > y$. $x < y$.
Proof: By theorem 9, definition 2 and definition 3. Qed.

Theorem 11: $x > y$ implies $y < x$.
Proof: For all x,y, we have $x > y$ iff there is a u such that $x = y + u$. Furthermore, we have $y < x$ iff there is a u such that $x = y+u$. So for all x, y, $x > y$ implies $y < x$. Qed.

Theorem 12: $x < y$ implies $y > x$.
Proof: We have $x < y$ iff there is a v such that $y = x + v$. Furthermore, we have $y > x$ iff there is a v such that $y = x + v$. So $x < y$ implies $y > x$. Qed.

Definition 4:
Define $x \geq y$ iff $x > y$ or $x = y$.

Definition 5:
Define $x \leq y$ iff $x < y$ or $x = y$.

Theorem 13: $x \geq y$ implies $y \leq x$.
Proof:
By theorem 11. Qed.

Theorem 14: $x \leq y$ implies $y \geq x$.
Proof:
By theorem 12. Qed.

Theorem 15: If $x < y$ and $y < z$ then $x < z$.
Proof: Assume $x < y$ and $y < z$. Then there is a v such that $y = x + v$. Furthermore, there is a u such that $z = y + u$. Then $z = (x + v) + u = x + (v + u)$. So there is a w such that $z = x + w$. So $x < z$. Qed.

Theorem 16: Let x, y, z be given. If $x \leq y$ and $y < z$ or $x < y$ and $y \leq z$ then $x < z$.
Proof:
By theorem 15. Qed.

Theorem 17: If $x \leq y$ and $y \leq z$ then $x \leq z$.
Proof:
By theorem 16. Qed.

Theorem 18: For all x,y, $x + y > x$.
Proof: For all x,y we have $x + y = x + y$. Qed.

Theorem 19: Let x, y, z be given. Then $x > y$ implies $x + z > y + z$, $x = y$ implies $x + z = y + z$ and $x < y$ implies $x + z < y + z$.
Proof:
Let z be given.
If $x > y$, then $x = y + u$, so $x + z = (y + u) + z = (u + y) + z = u + (y + z) = (y + z) + u$, i.e. $x + z > y + z$.
If $x = y$ then clearly $x + z = y + z$.
If $x < y$, then $y > x$, i.e. $y + z > x + z$, i.e. $x + z < y + z$. Qed.

Theorem 20: Let x, y, z be given. Then $x + z > y + z$ implies $x > y$, $x + z = y + z$ implies $x = y$ and $x + z < y + z$ implies $x < y$.
Proof:
By theorem 19. Qed.

Theorem 21: If $x > y$ and $z > u$ then $x + z > y + u$.
Proof:
Assume $x > y$ and $z > u$. Then by theorem 19 $x + z > y + z$. Then $y + z = z + y > u + y = y + u$. So $x + z > y + u$. Qed.

Theorem 22: Let x, y, z, u be given. If $x \geq y$ and $z > u$ or $x > y$ and $z \geq u$ then $x + z > y + u$.
Proof:
By theorem 19 and theorem 22. Qed.

Theorem 23: If $x \geq y$ and $z \geq u$ then $x + z \geq y + u$.
Proof:
Trivial. Qed.

Theorem 24: For all x, we have $x \geq 1$.

Proof:
Fix x. Then $x = 1$ or $x = u' = u + 1 > 1$. Qed.

Theorem 25: $y > x$ implies $y \geq x + 1$.
Proof:
Assume $y > x$. Then $y = x + u$. $u \geq 1$, i.e. $y \geq x + 1$. Qed.

Theorem 26: $y < x + 1$ implies $y \leq x$.
Proof:
Assume for a contradiction that $y < x + 1$ and $\neg y \leq x$. Then $y > x$. So by theorem 25 $y \geq x + 1$. Contradiction. Qed.

Definition 6:
Define $*$ recursively:
$x * 1 = x$.
$x * y' = (x * y) + x$.

Lemma 28a: For all y, $1 * y = y$.
Proof:
By definition 6, $1 * 1 = 1$.
Suppose $1 * y = y$. Then by definition 6, $1 * y' = (1 * y) + 1 = y + 1 = y'$.
Thus by induction, for all y $1 * y = y$. Qed.

Lemma 28b: For all x,y, $x' * y = (x * y) + y$.
Proof:
Fix x. Then $x' * 1 = x' = (x * 1)' = (x * 1) + 1$ by definition 6.
Suppose $x' * y = (x * y) + y$. Then by definition 6 $x' * y' = (x' * y) + x' = ((x * y) + y) + x' = (x * y) + (y + x') = (x * y) + (x' + y) = (x * y) + (x + y)' = (x * y) + (x + y') = ((x * y) + x) + y' = (x * y') + y'$.
Thus by induction, for all y $x' * y = (x * y) + y$. Qed.

Theorem 29: For all x, y, $x * y = y * x$.
Proof:
Fix y. Now $y * 1 = y$. By lemma 28a, $1 * y = y$, so $y * 1 = 1 * y$.
Now suppose $x * y = y * x$. Then $(x * y) + y = (y * x) + y = y * x'$. By lemma 28b, $x' * y = (x * y) + y$, so $x' * y = y * x'$.
Thus by induction, for all x $x * y = y * x$. Qed.

Theorem 30: For all x, y, z, $x * (y + z) = (x * y) + (x * z)$.
Proof:
Fix x, y. $x * (y + 1) = x * y' = (x * y) + x = (x * y) + (x * 1)$.
Now suppose $x * (y + z) = (x * y) + (x * z)$. Then $x * (y + z') = x * ((y + z)') = (x * (y + z)) + x = ((x * y) + (x * z)) + x = (x * y) + ((x * z) + x) = (x * y) + (x * z')$.
Thus by induction, for all z $x * (y + z) = (x * y) + (x * z)$. Qed.

Theorem 31: For all x, y, z, $(x * y) * z = x * (y * z)$.
Proof:
Fix x, y. Then $(x * y) * 1 = x * y = x * (y * 1)$.
Now suppose $(x*y)*z = x*(y*z)$. Then by theorem 30, $(x*y)*z' = ((x*y)*z)+(x*y) = (x * (y * z)) + (x * y) = x * ((y * z) + y) = x * (y * z')$.
Thus by induction, for all z $(x * y) * z = x * (y * z)$. Qed.

Theorem 32: For all z, $x > y$ implies $x * z > y * z$, $x = y$ implies $x * z = y * z$ and $x < y$ implies $x * z < y * z$.
Proof:
Let z be given.
If $x > y$, then $x = y + u$, i.e. $x * z = (y + u) * z = (y * z) + (u * z) > y * z$.
If $x = y$, then clearly $x * z = y * z$.
If $x < y$, then $y > x$, i.e. $y * z > x * z$, i.e. $x * z < y * z$. Qed.

Theorem 33: $x * z > y * z$ implies $x > y$, $x * z = y * z$ implies $x = y$ and $x * z < y * z$ implies $x < y$.
Proof:
By theorem 32 and theorem 10. Qed.

Theorem 34: If $x > y$ and $z > u$, then $x * z > y * u$.
Proof:
Suppose $x > y$ and $z > u$. By theorem 32, $x*z > y*z$ and $y*z = z*y > u*y = y*u$, i.e. $x * z > y * u$. Qed.

Theorem 35: If $x \geq y$, $z > u$ or $x > y$, $z \geq u$, then $x * z > y * z$.
Proof:
By theorem 32 and theorem 34. Qed.

Theorem 36: If $x \geq y$ and $z \geq u$, then $x * z \geq y * u$.
Proof:
By theorem 35. Qed.